Geometrical Nature of Mass and The Fifth Dimension of Kaluza-Klein Theory

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Abstract

The purpose of this paper is to show that formulation of general-relativistic kinetic theory in Kaluza-Klein context leads to interpretation of the fifth dimension as proper time. The rest mass of a particle is identified with the momentum conjugated to the fifth coordinate. It is shown that the mass shell constraint is a dynamical restriction following from the form of total action as a sum of the integral over the base manifold and the integral over the cotangent bundle. Based on this form of action the applicability of the differentiable manifold model is also briefly discussed[1].

1 Introduction

The physical nature of the extra dimension introduced in Kaluza-Klein unified theory of electromagnetism and gravitation has been the subject of intensive research since the very introduction of the theory. Numerous attempts to explain why classically observed spacetime is apparently four-dimensional had to either invoke the condition of cylindricity, due to Klein, which compactified the extra dimension down to the scale inaccessible to classical physics, or to abandon Einstein's conceptual framework of General Relativity replacing it with that of projective geometry. In this paper we shall not discuss various mechanisms for compactification, nor shall we develop the projective approach. It will suffice to mention that most of the difficulties here originated from attempts to give purely spatial or purely temporal interpretation of the extra dimension. Instead, we shall take the fifth dimension at its "face value", i.e. pursue what is usually referred to as

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"non-compactified approach", whereby all physical fields are allowed to depend explicitly on the fifth coordinate x^4 and no assumption on the topology of the manifold is made, except perhaps the usual requirement of it being paracompact to make useful operations such as integration meaningful.

It is assumed that the reader is familiar with the basic ideas of Kaluza's mechanism of unification of the gravitational and electromagnetic fields in a spacetime with five dimensions. For the review of the current state of Kaluza-Klein theory viewed in general-relativistic rather than particle physics context, see [1] and references therein.

The idea of the existence of deep connection between rest mass and the fifth dimension is due to Wesson and his collaborators ([1]). It is the main purpose of this paper to provide confirmation of this link as apparent in the domain of kinetic theory, naturally extended into the five-dimensional case. The fifth coordinate x^4 will be seen to map naturally to the proper time τ and the conjugated momentum p_4 corresponds to the rest mass m:

$$(x^4, p_4) \mapsto (\tau, m) \tag{1}$$

In this aspect we radically differ from the original approach of Wesson et. al. but nevertheless the fundamental connection to the rest mass (proposed by Wesson) is retained.

2 Dynamics

Let us consider dynamics of a test particle in the context of Einstein's General Relativity. General-relativistic form of kinetic theory is well-known (see [6], [7], [8]) and we shall only present here the main concepts and equations. By *spacetime* one assumes a connected four-dimensional time-oriented Lorentzian manifold with countable basis ([9], p. 51) denoted by $(\mathcal{M}, g)$, where $\mathcal{M}$ is the set of events and g metric of Lorentzian signature $(-,+,+,+)$: $g \in \mathrm{Lor}(\mathcal{M})$. The trajectory of a particle is modelled by a smooth curve: $\gamma : \mathbb{R}^1 \to \mathcal{M}$, obeying the equations of geodesics:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} = 0 \tag{2}$$

or in covariant form:

$$\nabla_X X = 0 \tag{3}$$

which can be obtained by varying the action functional of familiar form:

$$S[x^\mu(\tau)] = -\int_{\tau_A}^{\tau_B} \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\, d\tau \tag{4}$$

For our purposes it is more convenient to rewrite this second order system of n equations as a first order system of $2n$ equations, by using natural coordinates in the cotangent bundle $T^*\mathcal{M}$:

$$\begin{aligned}\frac{dx^\mu}{ds} &= \frac{1}{m}g^{\mu\nu}p_\nu \\ \frac{dp_\mu}{ds} &= -\frac{1}{2m}\frac{\partial g^{\lambda\rho}}{\partial x^\mu}p_\lambda p_\rho\end{aligned} \tag{5}$$

The base phase space of the system is the cotangent bundle $T^*\mathcal{M}$ over Lorentz manifold $(\mathcal{M}, g)$. The phase manifold $T^*\mathcal{M}$ has a natural symplectic structure that allows us to write the geodesic equations in hamiltonian form:

$$\begin{aligned}\frac{dx^\mu}{ds} &= \frac{\partial \mathcal{H}}{\partial p_\mu} \\ \frac{dp_\mu}{ds} &= -\frac{\partial \mathcal{H}}{\partial x^\mu}\end{aligned} \tag{6}$$

with the Hamiltonian function represented by:

$$\mathcal{H}(x,p) = \begin{cases} \frac{1}{2m}g^{\mu\nu}(x)p_\mu p_\nu + \frac{m}{2} & m > 0, \\ \frac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu & m = 0 \end{cases} \tag{7}$$

The submanifold (sometimes called *unit sphere*):

$$\Sigma_m(\mathcal{M}) = \left\{(x,p) \in T^*\mathcal{M} \mid g(p,p) + m^2 = 0\right\} = \Sigma_m^+(\mathcal{M}) \cup \Sigma_m^-(\mathcal{M}) \tag{8}$$

is called a *mass shell* and is dynamically invariant with respect to the one-parametric Lie group of local diffeomorphisms of the base phase space generated by the phase flow. This means that if initial position (x, p) lies on $\Sigma_m(\mathcal{M})$ then the whole trajectory lies on $\Sigma_m(\mathcal{M})$. Moreover, the above statement is true if $\Sigma_m(\mathcal{M})$ is replaced by upper or lower connected component $\Sigma_m^\pm(\mathcal{M})$ respectively. Interpreting, the objects moving on $\Sigma_m^-(\mathcal{M})$ as *antiparticles* we can say that dynamical invariance of each connected component of the mass shell means that there can be no transitions between particles and antiparticles (in classical regime, this changes when Wigner functions come into play in quantum regime).

Matter is described by a real-valued smooth function defined on a cotangent bundle $f : T^*\mathcal{M} \to \mathbb{R}$ which satisfies kinetic equation.

$$g^{\mu\nu}p_\nu\frac{\partial f}{\partial x^\mu} = \frac{1}{2}\frac{\partial g^{\lambda\rho}}{\partial x^\mu}p_\lambda p_\rho\frac{\partial f}{\partial p_\mu} \tag{9}$$

It is interesting to note that dynamical equations for $g^{\mu\nu}(x)$ and $f(x,p)$ can be derived from the action $\mathcal{S} = \mathcal{S}[g^{\mu\nu}(x), f(x,p)]$:

$$\mathcal{S} = \frac{1}{8\pi}\int\limits_{\mathcal{M}}(\mathcal{R}-2\Lambda)\sqrt{-g}\,d^nx - \int\limits_{\mathcal{M}}\sqrt{-g}\,d^nx\int\limits_{T_x^*\mathcal{M}}\frac{d^np}{\sqrt{-g}}f(x,p)g^{\mu\nu}(x)p_\mu p_\nu \quad (10)$$

This can also be written in the form of a sum of the purely geometric action $\mathcal{S}_H[g^{\mu\nu}(x)]$ (Hilbert action) and contribution of the matter $\mathcal{S}_M[g^{\mu\nu}(x), f(x,p)]$:

$$\mathcal{S}[g^{\mu\nu}(x)] = \mathcal{S}_H[g^{\mu\nu}(x)] + \mathcal{S}_M[g^{\mu\nu}(x), f(x,p)] \quad (11)$$

where each of the contributions is considered as a functional in the appropriate domain:

$$\mathcal{S}_H : \mathrm{Lor}(\mathcal{M}) \to \mathbb{R} \quad (12)$$

$$\mathcal{S}_M : \mathrm{Lor}(\mathcal{M}) \times \mathcal{F}(T^*\mathcal{M}) \to \mathbb{R} \quad (13)$$

where by $\mathcal{F}(T^*\mathcal{M})$ we have denoted the algebra of smooth functions defined on the cotangent bundle over $\mathcal{M}$ and $\mathrm{Lor}(\mathcal{M})$ is the class of all smooth Lorentzian metrics on a given differentiable manifold $\mathcal{M}$. It is worth noting that the expression for action functional is a sum of the integral over the base manifold $\mathcal{M}$ and of the integral over the cotangent bundle $T^*\mathcal{M}$. Due to non-compactness of the cotangent bundle one cannot transform the Hilbert action into the integral over $T^*\mathcal{M}$ - the problem is also known as the ultraviolet divergence. Perhaps this is an indication of the limitation of the manifold model for representing the spacetime. Indeed, differentiable manifold $x \in \mathcal{M}$ "locally looks" like $\mathbb{R}^n$ which means the cotangent space $T_x^*(\mathcal{M})$ at each point $x \in \mathcal{M}$ is an n-dimensional vector space and thus isomorphic to $\mathbb{R}^n$, in which arbitrarily large values of coordinates are allowed. Physically, this is rather unrealistic because infinite values of energy, momentum and proper mass (which correspond to the coordinates in $T_x^*(\mathcal{M})$) are unachievable. When one accelerates a particle to larger and larger values of the energy the structure of the manifold itself will be affected by the curvature produced by such energies. In other words, a "test particle with infinite energy" is no longer a "test particle". Looks like we need to replace the manifold with something better suited as a model of physical spacetime.

Varying $S[g^{\mu\nu}(x), f(x,p)]$ over $g^{\mu\nu}(x)$ and over $f(x,p)$ independently we obtain:

$$\frac{\delta S}{\delta g^{\mu\nu}(x)} = \frac{1}{8\pi}(\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu}) - \int\limits_{T_x^*\mathcal{M}}\frac{d^np}{\sqrt{-g}}f(x,p)p_\mu p_\nu \quad (14)$$

$$\frac{\delta S}{\delta f(x,p)} = -g^{\mu\nu}(x)p_\mu p_\nu \tag{15}$$

So, varying over the metric $g^{\mu\nu}$ yields Einstein equations (in n dimensions) and varying over the distribution function $f(x,p)$ dictates that all physical variables are to be supported on the null mass shell $p^2 = 0$. If we set $n = 4$ we obtain Lorentzian manifold filled with massless $m = 0$ gas in the context of Einstein's General Relativity. If we set $n = 5$ we obtain Kaluza-Klein generalization which includes electromagnetic ($U(1)$ gauge) and scalar field. Furthermore, we interpret the fifth dimension coordinate and momentum as proper time and the rest mass of the non-geometric substrate described by distribution function $f(x,p)$:

$$(x^4, p_4) \mapsto (\tau, m) \tag{16}$$

To see that such interpretation is indeed possible all one needs to do is to write down the geodesic equation in this framework and see how it maps naturally to equations (5) when the $4+1$ split is made. Such technicality is left as a simple exercise to the curious reader.

3 Future Developments

Several limitations are immediately apparent in the framework described above. First of all, even in four-dimensional spacetime it is well-known how to obtain quantum kinetic theory. One uses Wigner function and the formalism of cotangent bundles and is able to calculate explicitly quantum corrections to the classical distribution function to any order of $\hbar$. This formalism is described in [2]-[5]. It is not entirely obvious how to fit this formalism into the five-dimensional framework of Kaluza-Klein.

Also, we are reminded that the fundamental ingredient missing from the General Relativity of 1915 is still missing in the unified field theory of Kaluza-Klein. Namely, we have fixed the details of metric characterization of the manifold but left the topology almost entirely arbitrary. By "almost entirely" I mean that, of course, the very existence of Lorentzian metric implies certain restrictions of topological character but there is no mechanism of selection of a preferred topology compatible with a given metric $g_{\mu\nu}$.

What would be most interesting is to generalize the form of the action functional (4) on arbitrary fibre bundle or, alternatively, find a way to control the change of topology of $\mathcal{M}$ by imposing some additional structure on $T^*\mathcal{M}$ and modify the form of action accordingly, i.e. as an invariant with respect to such new structure.

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